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# ENERGY BOUNDS FOR THE SPINLESS SALPETER EQUATION

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## Abstract

We study the spectrum of the Salpeter Hamiltonian  $H = \beta\sqrt{m^2 + \mathbf{p}^2} + V(r)$ , where  $V(r)$  is an attractive central potential in three dimensions. If  $V(r)$  is a convex transformation of the Coulomb potential  $-1/r$  and a concave transformation of the harmonic-oscillator potential  $r^2$ , then upper and lower bounds on the discrete eigenvalues of  $H$  can be constructed, which may all be expressed in the form

$$E = \min_{r>0} \left[ \beta\sqrt{m^2 + \frac{P^2}{r^2}} + V(r) \right]$$

for suitable values of  $P$  here provided. At the critical point  $r = \hat{r}$  the relative growth to the Coulomb potential  $h(r) = -1/r$  must be bounded by  $dV/dh < 2\beta/\pi$ .

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# I Introduction

We study the (semirelativistic) “spinless-Salpeter” Hamiltonian

$$H = \beta\sqrt{m^2 + \mathbf{p}^2} + V(r) , \quad \beta > 0 , \quad (1)$$

in which  $V(r)$  is a central potential in three spatial dimensions. The eigenvalue equation of this operator is called the “spinless Salpeter equation.” This equation of motion arises as a well-defined standard approximation to the Bethe–Salpeter formalism [1] for the description of bound states within a (relativistic) quantum field theory and is arrived at by the following simplifying steps:

1. Eliminate all timelike variables by assuming the Bethe–Salpeter kernel that describes the interactions between the bound-state constituents to be static, i.e., instantaneous; the result of this reduction step is called the “instantaneous Bethe–Salpeter equation” or the “Salpeter equation” [2].
2. Neglect the spin of the bound-state constituents, assume the Bethe–Salpeter kernel to be of convolution type (as is frequently the case), and consider merely positive-energy solutions  $\psi$ , in order to arrive at the so-called “spinless Salpeter equation”  $H\psi = E\psi$ , with a Hamiltonian  $H$  of the form (1). (For two particles, this form of the Hamiltonian  $H$  holds only for equal masses  $m$  of the bound-state constituents.)

(For a more detailed account of the reduction of the Bethe–Salpeter equation to the spinless Salpeter equation, consult, e.g., the introductory sections of Refs. [3, 4].) This wave equation describes the bound states of spin-zero particles (scalar bosons) as well as the spin-averaged spectra of the bound states of fermions.

In this paper we consider potentials which are at the same time convex transformations  $V(r) = g(h(r))$  of the Coulomb potential  $h(r) = -1/r$  and concave transformations of the harmonic-oscillator potential  $h(r) = r^2$ . The reason for this is that spectral information is known for these two “basis” potentials  $h(r)$ . Thus the class of potentials is those  $V(r)$  that have a dual representation

$$V(r) = g^{(1)}\left(-\frac{1}{r}\right) = g^{(2)}(r^2) ,$$

in which  $g^{(1)}$  is convex ( $g^{(1)''} > 0$ ) and  $g^{(2)}$  is concave ( $g^{(2)''} < 0$ ). An example of a potential in this class is

$$V(r) = -\frac{c_1}{r} + c_2 \ln r + c_3 r + c_4 r^2 , \quad (2)$$

where the coefficients  $\{c_i\}$  are not negative and are not all zero. Thus tangent lines to the transformation function  $g(h)$  are of the form  $ah + b$  and are either Coulomb potentials lying below  $V$ , or harmonic-oscillator potentials lying above  $V$ . This geometrical idea is the basis for our approach to the spectral problem posed by  $H$ . We recall the application of this idea to the (nonrelativistic) Schrödinger problem in Sec. III. The general envelope formalism for the derivation of upper and lower bounds on the eigenvalues of the semirelativistic Salpeter Hamiltonian  $H$  of Eq. (1) is established in Sec. IV.

It is fundamental to our method that we first know something about the spectrum of  $H$  in those cases where  $V(r)$  is one of the basis potentials, i.e., the Coulomb and the harmonic oscillator. These two spectra are discussed in Sec. II below. In Sec. V we look at the example of the Coulomb-plus-linear potential.

## II The Coulomb and harmonic-oscillator potentials

### II.1 Scaling behaviour

Since the two basis potentials are both pure powers, it is helpful first to determine what can be learnt about the corresponding eigenvalues by the use of standard scaling arguments. By employing a wave function  $\phi(cr)$  depending on a scale variable  $c > 0$ , we find the following scaling rule for the eigenvalues corresponding to attractive pure power potentials  $v \operatorname{sgn}(q)r^q$ . The Hamiltonian

$$H = \beta \sqrt{m^2 + \mathbf{p}^2} + v \operatorname{sgn}(q)r^q$$

has the (energy) eigenvalues  $E(v, \beta, m)$ , where

$$E(v, \beta, m) = \beta m E\left(\frac{v}{\beta m^{1+q}}, 1, 1\right), \quad q \geq -1.$$

The scaling behaviour described by the above formula allows us to consider the one-particle, unit-mass special case  $m = \beta = 1$  initially, that is to say, to work w.l.o.g. with the operator

$$H = \sqrt{1 + \mathbf{p}^2} + v \operatorname{sgn}(q)r^q.$$

### II.2 Coulomb potential

In the case of the Coulomb potential  $V(r) = -v/r$  it is well known [5] that the Hamiltonian  $H$  has a Friedrichs extension provided the coupling constant  $v$  is not too large. Specifically, it is necessary in this case that  $v$  is smaller than a critical value  $v_c$  of the coupling constant:

$$v < v_c = \frac{2}{\pi}.$$

With this restriction, a lower bound to the bottom of the spectrum is provided by Herbst's formula

$$E_0 \geq \sqrt{1 - (\sigma v)^2}, \quad \sigma \equiv \frac{\pi}{2}. \quad (3)$$

By comparing the spinless Salpeter problem to the corresponding Klein–Gordon equation, Martin and Roy [6] have shown that if the coupling constant is further restricted by  $v < \frac{1}{2}$ , then an improved lower bound is provided by the expression

$$E_0 \geq \sqrt{\frac{1 + \sqrt{1 - 4v^2}}{2}}, \quad v < \frac{1}{2}. \quad (4)$$

It turns out that our lower-bound theory has a simpler form when the Coulomb eigenvalue formula has the form of Eq. (3) rather than that of Eq. (4). For this reason, we have derived from Eq. (4) a family of Coulomb bounds which, by rather elementary methods, is found to read

$$E_0 \geq \sqrt{1 - (\sigma v)^2}, \quad v \leq \frac{\sqrt{\sigma^2 - 1}}{\sigma^2} < \frac{1}{2}. \quad (5)$$

All these (lower) bounds are slightly weaker than the Martin–Roy bound (4) but above the Herbst bound (3). We note that these functions of the coupling constant  $v$  are all monotone and *concave*.

### II.3 Harmonic-oscillator potential

In the case of the harmonic-oscillator potential, i.e.,  $V(r) = vr^2$ , much more is known [7, 8]. In momentum-space representation the operator  $\mathbf{p}$  becomes a  $c$ -variable and thus, from the spectral point of view, the Hamiltonian  $H = \sqrt{1 + \mathbf{p}^2} + vr^2$  is equivalent to the Schrödinger operator

$$H = -v\Delta + \sqrt{1 + r^2}. \quad (6)$$

Since the potential  $V(r)$  increases without bound, the spectrum of  $H$  is entirely discrete [9]. We denote its eigenvalues by  $\mathcal{E}_{n\ell}(v)$ , where  $n = 1, 2, 3, \dots$  “counts” the radial states in each angular-momentum subspace labelled by  $\ell = 0, 1, 2, \dots$ . Below we shall either approximate the eigenvalues  $\mathcal{E}(v)$  analytically or assume them to be known numerically. The eigenvalues  $\mathcal{E}(v)$  of such Schrödinger operators are *concave* functions of the coupling constant  $v$  [10, 11].

### II.4 The spectral comparison theorem

For the class of interaction potentials given by (2) with the coefficient of the Coulombic term not too large, that is, for all potentials which satisfy the constraint  $\lim_{r \rightarrow 0} r^2 V'(r) < 2\beta/\pi$ , the semirelativistic Salpeter Hamiltonian  $H$  is bounded below and is essentially self-adjoint [5]. Consequently, the discrete spectrum of  $H$  is characterized variationally [9] and it follows immediately from this that, if we compare two such Hamiltonians  $H$  having the potentials  $V^{(1)}(r)$  and  $V^{(2)}(r)$ , respectively, and we know that  $V^{(1)}(r) < V^{(2)}(r)$ , then we may conclude that the corresponding discrete eigenvalues  $E_{n\ell}$  satisfy the inequalities  $E_{n\ell}^{(1)} < E_{n\ell}^{(2)}$ . We shall refer to this fundamental result as the “spectral comparison theorem.” In the more common case of nonrelativistic dynamics, i.e., for a (nonrelativistic) kinetic term of the form  $\beta\mathbf{p}^2/2m$  in the Hamiltonian  $H$ , a constraint similar to the above would hold for the coefficient of a possible additional (attractive)  $-1/r^2$  term in the potential  $V(r)$ .

## III General envelope theory of Schrödinger operators

In nonrelativistic envelope theory [11, 12, 13], if the potential  $V$  is a smooth transformation  $V(r) = g(\text{sgn}(q)r^q)$  of the power-law potential  $\text{sgn}(q)r^q$ , the eigenvalues of  $H = -\Delta + V(r)$  are approximated by

$$E_{n\ell} \approx \min_{r>0} \left[ \frac{P_{n\ell}^2(q)}{r^2} + V(r) \right]. \quad (7)$$

The numbers  $P_{n\ell}(q)$  can be derived from the eigenvalues of  $-\Delta + \text{sgn}(q)r^q$  [13]. If  $g$  is convex ( $g'' > 0$ ), Eq. (7) yields lower bounds; if  $g$  is concave ( $g'' < 0$ ), the results are upper bounds.

As an immediate application we consider the (nonrelativistic) Schrödinger Hamiltonian (6) for the Salpeter harmonic-oscillator problem (1). Here we have  $H = -v\Delta + V(r)$ , with  $V(r) = \beta\sqrt{m^2 + r^2}$ ; hence, the potential is a convex transformation of a linear potential and a concave transformation of a harmonic-oscillator potential. We conclude therefore from (7):

$$\min_{r>0} \left[ v \frac{P_{n\ell}^2(1)}{r^2} + \beta\sqrt{m^2 + r^2} \right] \leq \mathcal{E}_{n\ell}(v) \leq \min_{r>0} \left[ v \frac{P_{n\ell}^2(2)}{r^2} + \beta\sqrt{m^2 + r^2} \right]; \quad (8)$$

the numbers  $P_{n\ell}(1)$  are given in Table 1 of Ref. [14], and  $P_{n\ell}(2) = 2n + \ell - \frac{1}{2}$ . Interestingly the upper and lower bounds (8) are equivalent to the corresponding bounds obtained in Ref. [8]; however, these earlier specific bounds were not derived as part of a general theory.

## IV General envelope theory of Salpeter Hamiltonians

Let us now turn to our main topic and consider the spinless-Salpeter Hamiltonian of Eq. (1),

$$H = \beta\sqrt{m^2 + \mathbf{p}^2} + V(r) ,$$

and its eigenvalues  $E$ . We shall assume that the potential  $V(r)$  is a smooth transformation  $V(r) = g(h(r))$  of another potential  $h(r)$  and that  $g$  has definite convexity so that we obtain bounds to the energy eigenvalues  $E$ . We suppose that the “basis” potential  $h(r)$  generates a “tangential” Salpeter problem

$$\mathcal{H} = \beta\sqrt{m^2 + \mathbf{p}^2} + vh(r) ,$$

for which the eigenvalues  $e(v)$ , or bounds to them, are known. We shall follow here as closely as possible the development in Refs. [11, 12, 13] for the corresponding Schrödinger problem. We express our results in the form of two theorems.

1. The approximations we shall use from Sec. II, regarded as functions of the coupling  $v$ , and also the (unknown) energy functions  $e(v)$  of the “tangential” Salpeter problem are all *concave*:  $e''(v) < 0$ . The latter result represents the principal claim of Theorem 1.
2. In Theorem 2 we begin by using an envelope representation for the potential  $V(r)$  and then demonstrate that all the spectral formulas that follow possess a certain structure.

Finally, as an application, we specialize to the case of pure power-law “basis” potentials  $h(r)$  and, more particularly, to the Coulomb potential and the harmonic-oscillator potential for which, at this time, we have spectral information (cf. the discussions in Secs. II.2 and II.3).

### IV.1 Convexity of the energy function

We begin by proving

**Theorem 1** *The function  $e(v)$  is concave, that is,  $e''(v) < 0$ .*

**Proof** Suppose the exact eigenvalue and (normalized) eigenvector for the problem posed by  $\mathcal{H} = \beta\sqrt{m^2 + \mathbf{p}^2} + vh(r)$  are  $e(v)$  and  $\psi(v, r)$ . By differentiating  $(\psi, \mathcal{H}\psi)$  with respect to  $v$  we find  $e'(v) = (\psi, h\psi)$ . If we now apply  $\psi(v, r)$  as a trial vector to estimate the energy of the operator  $\beta\sqrt{m^2 + \mathbf{p}^2} + uh(r)$ , in which  $v$  has been replaced by  $u$ , we obtain an upper bound to  $e(u)$  which may be written in the form  $e(u) \leq e(v) + (u - v)e'(v)$ . This inequality tells us that the function  $e(u)$  lies beneath its tangents; that is to say,  $e(u)$  is *concave*.  $\square$

### IV.2 The principal envelope formula

With the help of Theorem 1 we are able to prove

**Theorem 2 (principal envelope formula)** *Suppose that the operator  $\beta\sqrt{m^2 + \mathbf{p}^2} + vh(r)$  has the exact lowest eigenvalue  $e(v)$ , and suppose that the operator  $\beta\sqrt{m^2 + \mathbf{p}^2} + g(h(r))$  has the exact lowest eigenvalue  $E$ . Then*

$$E \approx \mathcal{E} \equiv \min_{v>0} [e(v) - ve'(v) + g(e'(v))] . \quad (9)$$

*If  $g$  is concave (that is,  $g'' < 0$ ), then  $E \leq \mathcal{E}$ ; if  $g$  is convex (that is,  $g'' > 0$ ), then  $E \geq \mathcal{E}$ .*

**Proof** All the tangential potentials we shall employ have the form  $V^{(t)}(r) = a(t)h(r) + b(t)$ , where, as in the Schrödinger case, the coefficients  $a(t)$  and  $b(t)$  are given by

$$a(t) = \frac{V'(t)}{h'(t)} = g'(h(t)) , \quad b(t) = V(t) - a(t)h(t) = g(h(t)) - g'(h(t))h(t) ,$$

and  $r = t$  is the point of contact of the potential  $V(r)$  and its tangent  $V^{(t)}(r)$ . If, for the sake of definiteness, we assume that  $V = g(h)$  with  $g$  concave (i.e.,  $g'' < 0$ ), we obtain a family of upper bounds given by

$$E \leq \varepsilon(t) = e(a(t)) + b(t) .$$

The best of these is given by optimizing over  $t$ :

$$E \leq \varepsilon(\hat{t}) = e(a(\hat{t})) + b(\hat{t}) ,$$

where  $\hat{t}$ , the value of  $t$  which optimizes these bounds, is to be determined as the solution of

$$e'(g'(h(\hat{t}))) = h(\hat{t}) .$$

In the spirit of the Legendre transformation [15] we now consider another problem which has the same solution; this second problem is the one that provides us with our basic eigenvalue formula. We consider

$$\mathcal{E} = \min_{v>0} [e(v) - ve'(v) + g(e'(v))] ,$$

which is well defined since  $e(v)$  is concave. The solution has the critical point  $\hat{v} = g'(e'(\hat{v}))$ . If we now apply the correspondence  $h(\hat{t}) = e'(\hat{v})$ , it follows that the critical point  $\hat{v}$  becomes  $\hat{v} = g'(h(\hat{t}))$  and the tangential-potential coefficients  $a$  and  $b$  become

$$a(\hat{t}) = g'(e'(v)) = v , \quad b(\hat{t}) = g(e'(v)) - ve'(v) , \quad v = \hat{v} . \quad (10)$$

Meanwhile the original critical (energy) value is given by

$$\varepsilon(\hat{t}) = e(a(\hat{t})) + b(\hat{t}) = e(v) - ve'(v) + g(e'(v)) , \quad v = \hat{v} .$$

□

From the proof of Theorem 2 it follows immediately that, if the *exact* energy function  $e(v)$  corresponding to the basis potential  $h$  is not available, then, for  $g(h)$  concave, concave *upper* approximations  $e_u(v) > e(v)$  or, for  $g(h)$  convex, concave *lower* approximations  $e_l(v) < e(v)$  may be used instead of the exact energy function  $e(v)$  in the principal envelope formula (9). Then all the lower tangents will lie even lower and all the upper tangents will lie even higher. If  $g$  is convex, we obtain a lower bound; if  $g$  is concave, we obtain an upper bound; because of the concavity of  $e(v)$ , *this* extremum is a minimum in *both* cases. If we wish to use numerical solutions to the “basis” problem (generated by  $h(r)$ ), or if a completely new energy-bound expression becomes available, the principal envelope formula (9) is what would be used first.

Interestingly, in the formula (9) the tangential-potential apparatus is no longer evident; only the correct convexity is required. As in the Schrödinger case [11], once we have the basic result, the reformulation in terms of “kinetic potentials” is often useful: the kinetic potential  $\bar{h}(s)$  corresponding to some basis potential  $h(r)$  is given by the Legendre transformation [15]

$$\bar{h}(s) = e'(v) , \quad s = e(v) - ve'(v) .$$

Meanwhile the envelope approximation has the kinetic-potential expression  $\bar{V}(s) \approx g(\bar{h}(s))$ .

For both the Coulomb lower bounds (3) or (5) and the harmonic-oscillator upper bounds (8) which we have at present, we may express our general results in a special common form which will now be derived.

### IV.3 The Coulomb lower bound

We consider first the Coulomb lower bound in which we assume that the potential  $V(r)$  is a convex transformation  $V(r) = g(h(r))$  of the Coulomb potential  $h(r) = -1/r$ . According to Sec. II.2, in this case all the “lower”  $e_1(v)$  have been arranged—with the parameters  $\beta$  and  $m$  returned—in the form

$$e_1(v) = \beta m \sqrt{1 - \left(\frac{\sigma v}{\beta}\right)^2}.$$

From this it follows by elementary algebra that if we define a new optimization variable  $r$  by  $e'_1(v) = h(r) = -1/r$ , we have

$$e_1(v) - v e'_1(v) = \beta \sqrt{m^2 + \frac{P^2}{r^2}}, \quad P \equiv \frac{1}{\sigma}.$$

Consequently, the lower bound on the energy eigenvalues  $E$  of the spinless Salpeter equation becomes

$$E \geq \min_{r>0} \left[ \beta \sqrt{m^2 + \frac{P^2}{r^2}} + V(r) \right], \quad v < \beta v_P. \quad (11)$$

Here the boundary value  $v_P$  of the Coulomb coupling  $v$  is given, when simply determined by the requirement of boundedness from below of the operator (1), by the critical coupling  $v_c$ ,

$$v_P = v_c = \frac{2}{\pi},$$

and, when arising from the region of validity of our Coulomb-like family of lower bounds (5), via  $P = 1/\sigma$ , by

$$v_P = P \sqrt{1 - P^2} < \frac{1}{2}. \quad (12)$$

$\{P, v_P\}$  pairs may be easily generated from the upper bound on the coupling  $v$  in Eq. (12). The meaning of the Coulomb-coupling constraint is  $a(\hat{t}) < \beta v_P$ , where  $a$  is the coefficient in the tangential Coulomb potential given by (10).

### IV.4 The harmonic-oscillator upper bounds

Next, let us turn to the harmonic-oscillator upper bounds. Our main assumption is here that  $V(r) = g(r^2)$ , with  $g'' < 0$ . In this case the only difficulty is that the basis problem  $h(r) = r^2$  is equivalent to a Schrödinger problem whose solution  $\mathcal{E}_{n\ell}(v)$  is not known exactly. Following the discussion after the proof of Theorem 2, let us call the upper bound provided by Eq. (8)  $e_u(v)$  and let us introduce the shorthand notation  $P_{n\ell}(2) = 2n + \ell - \frac{1}{2} = P$ . Then we have the following parametric equations for  $e_u(v)$ :

$$e_u(v) = v \frac{P^2}{r^2} + \beta \sqrt{m^2 + r^2}, \quad v = \frac{\beta r^4}{2P^2 \sqrt{m^2 + r^2}}, \quad e'_u(v) = \frac{P^2}{r^2}.$$

By substituting these expressions into the fundamental envelope formula (9) we obtain the following upper bound on all the eigenvalues of the spinless-Salpeter problem with potential  $V(r) = g(r^2)$  and  $g'' < 0$ :

$$E_{n\ell} \leq \min_{r>0} \left[ \beta \sqrt{m^2 + \frac{P^2}{r^2}} + V(r) \right], \quad P = P_{n\ell}(2) = 2n + \ell - \frac{1}{2}. \quad (13)$$

## V The Coulomb-plus-linear (or “funnel”) potential

In order to illustrate the above results by a physically motivated example, let us apply these considerations to the Coulomb-plus-linear or (in view of its shape) “funnel” potential

$$V(r) = -\frac{c_1}{r} + c_2 r, \quad c_1 \geq 0, \quad c_2 \geq 0.$$

(This potential provides a reasonable overall description of the strong interactions of quarks in hadrons. For the phenomenological description of hadrons in terms of both nonrelativistic and semirelativistic potential models, see, e.g., Refs. [16, 17].) By choosing as basis potential  $h(r) = -1/r$ , we may write  $V(r) = g(h(r))$  with

$$g(h) = c_1 h - \frac{c_2}{h},$$

which is clearly a convex function of  $h < 0$ :  $g'' > 0$ . Thus the convexity condition is satisfied. However, we are not free to choose the coupling constants  $c_1$  and  $c_2$  as large as we please. It is immediately obvious that, for a particular  $\{P, v_P\}$  pair, we must in any case have  $c_1 < \beta v_P$ . For the full problem the coefficient  $c_2$  of the linear term will also be involved. The coupling  $v$  we are concerned about is given by (10). We have

$$v = g'(e'(v)) = \frac{\beta P^2}{r \sqrt{m^2 + \left(\frac{P}{r}\right)^2}} = c_1 + \frac{c_2}{h^2} = c_1 + c_2 r^2.$$

From this we obtain, for given values of the parameters  $m$  and  $\beta$  and for a given  $\{P, v_P\}$  pair, as a sufficient condition for  $v < \beta v_P$  the “Coulomb coupling constant constraint” on the two coupling strengths  $c_1$  and  $c_2$  in the funnel potential:

$$c_1 + \frac{P^2}{m^2} \left( \frac{P^2}{v_P^2} - 1 \right) c_2 < \beta v_P. \quad (14)$$

In the case  $\{P = 1/\sqrt{2}, v_P = 1/2\}$  and  $\beta = m = 1$  this condition reduces to  $c_1 + \frac{1}{2}c_2 < \frac{1}{2}$ . For Herbst’s lower bound (3), i.e.,  $P = v_P = v_c = 2/\pi$ , this constraint clearly yields  $c_1 < \beta v_P$ . There is no escaping this feature of all energy bounds involving the Coulomb potential: the constraint derives from the fundamental observation that the Coulomb coupling  $v$  must not be too large, so that the (relativistic) kinetic energy is able to counterbalance the Coulomb potential in order to maintain the Hamiltonian (1) with  $V(r) = -v/r$  bounded from below.

For example, if we seek the largest allowed value of the parameter  $P$  by solving Eqs. (12) and (14) together, we find that this largest  $P$  is given by

$$\frac{c_2 \sin^4 t}{\cos^2 t (\beta \sin t \cos t - c_1)} = m^2, \quad P \equiv \sin t. \quad (15)$$

For the Coulomb-plus-linear potential  $V(r) = -c_1/r + c_2 r$  under consideration, Fig. 1 shows the lower and upper bounds on the lowest energy eigenvalue  $E$  of the spinless Salpeter equation, given by the envelopes of the lower and upper families of tangential energy curves (11) and (13). In the case of the lower bound (11), we have used for each  $m$  the best possible

$P(m)$  provided by (15). As  $m \rightarrow 0$ , the “basis” Coulomb problem  $H = \beta\sqrt{m^2 + \mathbf{p}^2} - v/r$  has energy  $e(m) \rightarrow 0$ ; thus the Coulomb lower bound for a non-Coulomb problem becomes very weak for small values of  $m$ . Of course, Eq. (13) provides us with rigorous upper bounds for *every* energy level.

In order to get an idea of the location of the exact energy eigenvalues  $E$ , Fig. 1 also shows the ground-state energy curve  $E(m)$  obtained by the Rayleigh–Ritz variational technique [9] with the Laguerre basis states for the trial space defined in Ref. [18]. Strictly speaking, this energy curve represents only an upper bound to the precise eigenvalue  $E$ . However, from the findings of Ref. [18] the deviations of these Laguerre bounds from the exact eigenvalues may be estimated, for the superposition of 25 basis functions used here, to be of the order of 1 %.

## VI Summary and conclusion

In this analysis we have studied the discrete spectrum of semirelativistic “spinless-Salpeter” Hamiltonians  $H$ , defined in Eq. (1), by an approach which is based principally on convexity. We have at our disposal very definite information concerning, on the one hand, the bottom of the spectrum of  $H$  for the Coulomb potential,  $h(r) = -1/r$ , and, on the other hand, the entire spectrum of  $H$  for the harmonic-oscillator potential,  $h(r) = r^2$ . The class of potentials that are at the same time a convex transformation of  $-1/r$  and a concave transformation of  $r^2$  includes, for example, arbitrary linear combinations of Coulomb, logarithmic, linear, and harmonic-oscillator terms. The envelope technique applied here takes advantage of the fact that all “tangent lines” to the interaction potential  $V(r) = g(h(r))$  in  $H$  are potentials of the form  $ah(r) + b$ , and that, by convexity and the comparison theorem recalled in Subsec. II.4, the energy eigenvalues corresponding to these “tangent” potentials provide rigorous bounds to the unknown exact eigenvalues  $E$  of  $H$ . If  $e(v)$  denotes the energy function—or a suitable bound to it—corresponding to the problem posed by a “basis” potential  $vh(r)$ , where  $v$  is a positive coupling parameter, the envelopes of upper and lower families of energy curves may be found with the help of the “principal envelope formula”

$$E \approx \min_{v>0} [e(v) - ve'(v) + g(e'(v))] .$$

Here, a sign of approximate equality is used to indicate that, for a definite convexity of  $g(h)$ , the envelope theory yields lower bounds for convex  $g(h)$  and upper bounds for concave  $g(h)$ . With the above principal envelope formula at hand, all new spectral pairs  $\{h(r), e(v)\}$  which may become available at some future time can immediately be used to enrich our collection of energy bounds. If the basis potential  $h(r)$  is a pure power, these bounds can be written as

$$E_{n\ell} \approx \min_{r>0} \left[ \beta \sqrt{m^2 + \frac{P_{n\ell}^2}{r^2}} + V(r) \right] ,$$

where the numbers  $P_{n\ell}$  are obtained from the corresponding underlying basis problems. The power of this technique is illustrated, in Sec. V, by our application to the funnel potential,  $V(r) = -c_1/r + c_2r$ . For this problem, we have employed both the semirelativistic Coulomb and harmonic-oscillator problems to calculate, respectively, lower and upper bounds on the energy eigenvalues of the spinless Salpeter equation.

We expect that such results would provide bounds on the energy eigenvalues for general theoretical discussions, or be used as guides for more tightly focussed analytic or numerical studies of the spectra of semirelativistic “spinless-Salpeter” Hamiltonians.

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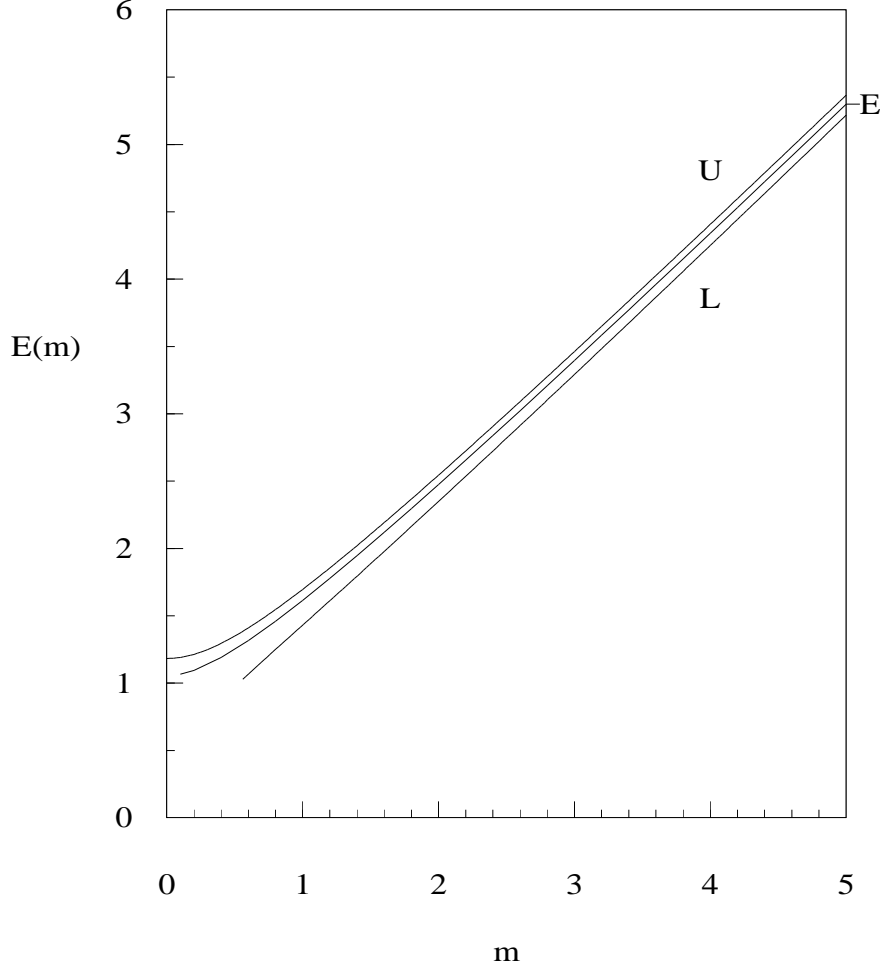


Figure 1: Lower bounds (L), according to (11), and upper bounds (U), according to (13), on the energy eigenvalue  $E$  of the ground state  $[(n, \ell) = (1, 0)]$  of the spinless Salpeter equation with a Coulomb-plus-linear potential  $V(r) = -c_1/r + c_2 r$ , for  $\beta = 1$ ,  $c_1 = 0.1$ , and  $c_2 = 0.25$ . The lower bound is given by the general result (11) with the “best”  $P(m)$  provided by (15). In order to satisfy the Coulomb coupling constraint (15), the mass  $m$  must fulfil  $m > \sqrt{5}/4$ . For comparison, a (very accurate) Rayleigh–Ritz variational upper bound  $E$  is depicted too.